MASS MINIMIZERS AND CONCENTRATION FOR NONLINEAR CHOQUARD EQUATIONS IN $\mathbb{R}^N$

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ABSTRACT. In this paper, we study the existence of minimizers to the following functional related to the nonlinear Choquard equation:

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|u|^2 - \frac{1}{2p}\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p$$

on $\widetilde{S}(c) = \{u \in H^1(\mathbb{R}^N)|\ \int_{\mathbb{R}^N} V(x)|u|^2 < +\infty,\ |u|_2 = c, c > 0\}$, where $N \geq 1$ $\alpha \in (0, N)$, $\frac{N+\alpha}{N} \leq p < \frac{N+\alpha}{(N-2)_+}$ and $I_\alpha : \mathbb{R}^N \to \mathbb{R}$ is the Riesz potential. We present sharp existence results for $E(u)$ constrained on $\widetilde{S}(c)$ when $V(x) \equiv 0$ for all $\frac{N+\alpha}{N} \leq p < \frac{N+\alpha}{(N-2)_+}$. For the mass critical case $p = \frac{N+\alpha+2}{N}$, we show that if $0 \leq V(x) \in L^\infty_{loc}(\mathbb{R}^N)$ and $\lim\limits_{|x|\to+\infty} V(x) = +\infty$, then mass minimizers exist only if $0 < c < c_* = |Q|_2$ and concentrate at the flattest minimum of V as c approaches c_* from below, where Q is a groundstate solution of $-\Delta u + u = (I_\alpha * |u|^{\frac{N+\alpha+2}{N}})|u|^{\frac{N+\alpha+2}{N}-2}u$ in $\mathbb{R}^N$.

Keywords: Choquard equation; Mass concentration; Normalized solutions; Sharp existence.
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1. INTRODUCTION

In this paper, we consider the following semilinear Choquard problem

$$-\Delta u - \mu u = (I_\alpha * |u|^p)|u|^{p-2}u, \quad x \in \mathbb{R}^N,\ \mu \in \mathbb{R} \tag{1.1}$$

where $N \geq 1$, $\alpha \in (0, N)$, $\frac{N+\alpha}{N} \leq p < \frac{N+\alpha}{(N-2)_+}$, here $\frac{N+\alpha}{(N-2)_+} = \frac{N+\alpha}{N-2}$ if $N \geq 3$ and $\frac{N+\alpha}{(N-2)_+} = +\infty$ if $N = 1, 2$. $I_\alpha : \mathbb{R}^N \to \mathbb{R}$ is the Riesz potential [23] defined as

$$I_\alpha(x) = \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi^{\frac{N}{2}}2^\alpha}\frac{1}{|x|^{N-\alpha}}, \quad \forall\ x \in \mathbb{R}^N\backslash\{0\}.$$

Problem (1.1) is a nonlocal one due to the existence of the nonlocal nonlinearity. It arises in various fields of mathematical physics, such as quantum mechanics, physics of laser beams, the physics of multiple-particle systems, etc. When $N = 3$, $\mu = -1$ and $\alpha = p = 2$, (1.1) turns to be the well-known Choquard-Pekar equation:

$$-\Delta u + u = (I_2 * |u|^2)u, \quad x \in \mathbb{R}^3, \tag{1.2}$$

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which was proposed as early as in 1954 by Pekar [22], and by a work of Choquard 1976 in a certain approximation to Hartree-Fock theory for one-component plasma, see [11, 13]. (1.1) is also known as the nonlinear stationary Hartree equation since if u solves (1.1) then $\psi(t,x) = e^{it}u(x)$ is a solitary wave of the following time-dependent Hartree equation

$$i\psi_t = -\Delta\psi - (I_\alpha * |\psi|^p)|\psi|^{p-2}\psi \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^N,$$

see [6, 18].

In the past years, there are several approaches to construct nontrivial solutions of (1.1), see e.g. [5, 11, 14, 15, 17, 18, 24] for $p = 2$ and [19, 20]. One of them is to look for a constrained critical point of the functional

$$I_p(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 - \frac{1}{2p}\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p \tag{1.3}$$

on the constrained L^2-spheres in $H^1(\mathbb{R}^N)$:

$$S(c) = \{u \in H^1(\mathbb{R}^N)|\ |u|_2 = c, c > 0\}.$$

In this way, the parameter $\mu \in \mathbb{R}$ will appear as a Lagrange multiplier and such solution is called a normalized solution. By the following well known Hardy-Littlewood-Sobolev inequality: For $1 < r, s < +\infty$, if $f \in L^r(\mathbb{R}^N)$, $g \in L^s(\mathbb{R}^N)$, $\lambda \in (0, N)$ and $\frac{1}{r} + \frac{1}{s} + \frac{\lambda}{N} = 2$, then

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{f(x)g(y)}{|x-y|^\lambda} \le C_{r,\lambda,N}|f|_r|g|_s, \tag{1.4}$$

we see that $I_p(u)$ is well defined and a C^1 functional. Set

$$I_p(c^2) = \inf_{u\in S(c)} I_p(u), \tag{1.5}$$

then minimizers of $I_p(c^2)$ are exactly critical points of $I_p(u)$ constrained on $S(c)$.

Normalized solutions for equation (1.2) have been studied in [11, 14]. In this paper, one of our purposes is to get a general and sharp result for the existence of minimizers for the minimization problem (1.5).

To state our main result, we first prove the following interpolation inequality with the best constant: For $\frac{N+\alpha}{N} < p < \frac{N+\alpha}{(N-2)_+}$,

$$\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p \le \frac{p}{|Q_p|_2^{2p-2}} \left(\int_{\mathbb{R}^N} |\nabla u|^2\right)^{\frac{Np-(N+\alpha)}{2}} \left(\int_{\mathbb{R}^N} |u|^2\right)^{\frac{N+\alpha-(N-2)p}{2}}, \tag{1.6}$$

where equality holds for $u = Q_p$, where Q_p is a nontrivial solution of

$$-\frac{Np-(N+\alpha)}{2}\Delta Q_p + \frac{N+\alpha-(N-2)p}{2}Q_p = (I_\alpha * |Q_p|^p)|Q_p|^{p-2}Q_p, \quad x \in \mathbb{R}^N. \tag{1.7}$$

In particular, $Q_{\frac{N+\alpha+2}{N}}$ is a groundstate solution, i.e. the least energy solution among all nontrivial solutions of (1.7). Moreover, when $p = \frac{N+\alpha+2}{N}$, all groundstate solutions of (1.7) have the same L^2-norm (see Lemma 3.2 below).

Recall in [12] that for $p=\frac{N+\alpha}{N}$, the following Hardy-Littlewood-Sobolev inequality with the best constant:

$$\int_{\mathbb{R}^N}(I_\alpha*|u|^{\frac{N+\alpha}{N}})|u|^{\frac{N+\alpha}{N}} \leq \frac{1}{|Q_{\frac{N+\alpha}{N}}|_2^{\frac{2(N+\alpha)}{N}}}\left(\int_{\mathbb{R}^N}|u|^2\right)^{\frac{N+\alpha}{N}} \tag{1.8}$$

with equality if and only if $u=Q_{\frac{N+\alpha}{N}}$, where $Q_{\frac{N+\alpha}{N}}=C\left(\frac{\eta}{\eta^2+|x-a|^2}\right)^{\frac{N}{2}}$, $C>0$ is a fixed constant, $a\in\mathbb{R}^N$ and $\eta\in(0,+\infty)$ are parameters.

Then our first result is as follows:

Theorem 1.1. *Assume that $N\geq 1$, $\alpha\in(0,N)$ and $\frac{N+\alpha}{N}\leq p<\frac{N+\alpha}{(N-2)_+}$.*

(1) *If $p=\frac{N+\alpha}{N}$, for any $c>0$,*

$$I_{\frac{N+\alpha}{N}}(c^2)=-\frac{N}{2(N+\alpha)}\left(\frac{c}{|Q_{\frac{N+\alpha}{N}}|_2}\right)^{\frac{2(N+\alpha)}{N}}$$

and $I_{\frac{N+\alpha}{N}}(c^2)$ has no minimizer.

(2) *If $\frac{N+\alpha}{N}<p<\frac{N+\alpha+2}{N}$, then $I_p(c^2)<0$ for all $c>0$, moreover, $I_p(c^2)$ has at least one minimizer for each $c>0$.*

(3) *If $p=\frac{N+\alpha+2}{N}$, let $c_*:=|Q_{\frac{N+\alpha+2}{N}}|_2$, then*

(i) $I_{\frac{N+\alpha+2}{N}}(c^2)=\begin{cases}0, & \text{if } 0<c\leq c_*,\\ -\infty, & \text{if } c>c_*;\end{cases}$

(ii) $I_{\frac{N+\alpha+2}{N}}(c^2)$ has no minimizer if $c\neq c_*$;

(iii) each groundstate of (1.7) is a minimizer of $I_{\frac{N+\alpha+2}{N}}(c_*^2)$.

(iv) there is no critical point for $I_{\frac{N+\alpha+2}{N}}(u)$ constrained on $S(c)$ for each $0<c<c_*$.

(4) *If $\frac{N+\alpha+2}{N}<p<\frac{N+\alpha}{(N-2)_+}$, then $I_p(c^2)$ has no minimizer for each $c>0$ and $I_p(c^2)=-\infty$.*

Remark 1.2. Theorem 1.1 can be seemed as a consequence of the results in Theorem 9 of [11] for $p=2$ and in Theorem 1 of [19]. However, we still state and prove Theorem 1.1 here by using an alternative method since our result is delicate and it provides a framework to our subsequent main considerations.

Remark 1.3.

(1) c_* is unique.

(2) Since the positive solution of (1.7) with $\alpha=p=2$ is uniquely determined up to translations see e.g. [3, 8, 10], it follows that if $N=4$ and $\alpha=2$, then up to translations, **the minimizer of $I_{\frac{N+\alpha+2}{N}}(c_*^2)$ is unique and there exists no critical point for $I_{\frac{N+\alpha+2}{N}}(u)$ constrained on $S(c)$ for each $c\neq c_*$.**

(3) For $N\geq 3$ and $\frac{N+\alpha+2}{N}<p<\frac{N+\alpha}{N-2}$, it has been proved in [9] that for each $c>0$, $I_p(u)$ has a mountain pass geometry on $S(c)$ and there exits a couple $(u_c,\mu_c)\in S(c)\times\mathbb{R}^-$ solution of (1.1) with $I_p(u_c)=\gamma(c)$, where $\gamma(c)$ denotes the mountain pass level on $S(c)$.

By Theorem 1.1, $p = \frac{N+\alpha+2}{N}$ is called L^2-critical exponent for (1.5). In order to get critical points under the mass constraint for such L^2-critical case, we add a nonnegative perturbation term to the right hand side of (1.3), i.e. considering the following functional:

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|u|^2 - \frac{N}{2(N+\alpha+2)}\int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{N+\alpha+2}{N}})|u|^{\frac{N+\alpha+2}{N}}, \quad (1.9)$$

where

$$V(x) \in L^\infty_{loc}(\mathbb{R}^N), \quad \inf_{x\in\mathbb{R}^N} V(x) = 0 \quad \text{and} \quad \lim_{|x|\to+\infty} V(x) = +\infty. \qquad (V_0)$$

Based on (V_0), we introduce a Sobolev space $\mathcal{H} = \{u \in H^1(\mathbb{R}^N)| \int_{\mathbb{R}^N} V(x)|u^2 < +\infty\}$ with its associated norm $\|u\|_{\mathcal{H}} = (\int_{\mathbb{R}^N}(|\nabla u|^2 + |u|^2 + V(x)|u|^2))^{\frac{1}{2}}$.

Theorem 1.4. *Assume that $N \geq 1$, $\alpha \in (0, N)$ and (V_0) holds. Set*

$$e_c = \inf_{u\in\widetilde{S}(c)} E(u), \qquad (1.10)$$

where $\widetilde{S}(c) = \{u \in \mathcal{H}|\ |u|_2 = c\}$. Let c_* be given in Theorem 1.1.

(1) If $0 < c < c_*$, then e_c has at least one minimizer and $e_c > 0$;

(2) Let $N - 2 \leq \alpha < N$ if $N \geq 3$ and $0 < \alpha < N$ if $N = 1, 2$, then for each $c \geq c_*$, e_c has no minimizer; Moreover, $e_c = \begin{cases} 0, & \text{if } c = c_* \\ -\infty, & \text{if } c > c_* \end{cases}$ and $\lim_{c\to(c_*)^-} e_c = e_{c_*}$.

We also concern the concentration phenomena of minimizers of e_c as c converges to c_* from below. Let u_c be a minimizer of e_c for each $0 < c < c_*$, then by (1.6) and Theorem 1.4, we see that $\int_{\mathbb{R}^N} V(x)|u_c|^2 \to 0$ as $c \to (c_*)^-$, i.e. u_c can be expected to concentrate at the minimum of $V(x)$. To show this fact, besides condition (V_0), we assume that there exist $m \geq 1$ distinct points $x_i \in \mathbb{R}^N$ and $q_i > 0$ $(1 \leq i \leq m)$ such that

$$\mu_i := \lim_{x\to x_i} \frac{V(x)}{|x - x_i|^{q_i}} \in (0, +\infty). \qquad (V_1)$$

Set

$$q := \max\{q_1, q_2, \cdots, q_m\}.$$

Let $\{c_k\} \subset (0, c_*)$ be a sequence such that $c_k \to c_*$ as $k \to +\infty$. Then Our main result is as follows:

Theorem 1.5. *Suppose that $N \geq 1$, $\alpha \in [N-2, N)$ if $N \geq 3$ and $\alpha \in (0, N)$ if $N = 1, 2$ and $(V_0)(V_1)$ hold. Then there exists a sequence $\{x_k\} \subset \mathbb{R}^N$ and a groundstate solution W_0 of the following equation*

$$-\Delta W_0 + W_0 = (I_\alpha * |W_0|^{\frac{N+\alpha+2}{N}})|W_0|^{\frac{N+\alpha+2}{N}-2}W_0, \quad x \in \mathbb{R}^N \qquad (1.11)$$

and

$$\lambda := \min_{1\leq i\leq m}\left\{\lambda_i|\ \lambda_i = \left(\frac{q_i}{2c_*^2}\mu_i \int_{\mathbb{R}^N} |x|^{q_i}|W_0(x)|^2\right)^{\frac{1}{q_i+2}}\right\}$$

such that up to a subsequence,

$$[1-(\frac{c_k}{c_*})^{\frac{2(\alpha+2)}{N}}]^{\frac{1}{q+2}\frac{N}{2}}u_{c_k}([1-(\frac{c_k}{c_*})^{\frac{2(\alpha+2)}{N}}]^{\frac{1}{q+2}}x+x_k)\to[(\frac{\alpha+2}{N})^{\frac{1}{q+2}}\lambda]^{\frac{N}{2}}W_0((\frac{\alpha+2}{N})^{\frac{1}{q+2}}\lambda x) \tag{1.12}$$

in $L^{\frac{2Ns}{N+\alpha}}(\mathbb{R}^N)$ *for* $\frac{N+\alpha}{N}\le s<\frac{N+\alpha}{(N-2)+}$ *as* $k\to+\infty$. *Moreover, there exists* $x_{j_0}\in\{x_i|\ \lambda_i=\lambda,\ 1\le i\le m\}$ *such that* $x_k\to x_{j_0}$ *as* $k\to+\infty$.

Remark 1.6. It has been proved in [19] that for $\alpha\in[N-2,N)$ if $N\ge3$ and $\alpha\in(0,N)$ if $N=1,2$, then each groundstate solution u of (1.11) satisfies that $\lim\limits_{|x|\to+\infty}|u(x)||x|^{\frac{N-1}{2}}e^{|x|}\in(0,+\infty)$. Hence $\lambda_i\in(0,+\infty)$.

The result in Theorem 1.5 is different from that in [16] studying the case $p<\frac{N+\alpha+2}{N}$, where one considered the concentration behavior of minimizers as $c\to+\infty$. The concentration phenomena have also been studied in [21] and [4] by considering semiclassical limit of the Choquard equation

$$-\varepsilon^2\Delta u+Vu=\varepsilon^{-\alpha}(I_\alpha*|u|^p)|u|^{p-2}u \quad \text{in } \mathbb{R}^N.$$

However, since the parameter is different, we need a different technique to obtain our result.

The main proof of Theorem 1.5 is based on optimal energy estimates of e_c and $\int_{\mathbb{R}^N}|\nabla u_c|^2$ for each minimizer u_c. The main idea to prove Theorem 1.5 comes from [7], which was restricted to the case of local nonlinearities. But due to the fact that our nonlinearity is nonlocal and that the assumption imposed on (V) is more general than that in [7], the method used in [7] can not be directly applied here. It needs some improvements and careful analysis. First, by choosing a suitable test function, we get that $0<e_c\le C_1[1-(\frac{c}{c_*})^{\frac{2(\alpha+2)}{N}}]^{\frac{q}{q+2}}$ as $c\to(c_*)^-$ for some constant $C_1>0$ independent of c. The lower bound now is not optimal. The method in [7] by using the perturbation term $\int_{\mathbb{R}^N}V(x)u^2$ to remove the local nonlinearity term does not work in our cases. To obtain an optimal lower bound, we notice that $\int_{\mathbb{R}^N}|\nabla u_c|^2\to+\infty$ as $c\to(c_*)^-$, moreover,

$$\lim_{c\to(c_*)^-}\frac{\frac{N}{N+\alpha+2}\int_{\mathbb{R}^N}(I_\alpha*|u_c|^{\frac{N+\alpha+2}{N}})|u_c|^{\frac{N+\alpha+2}{N}}}{\int_{\mathbb{R}^N}|\nabla u_c|^2}=1.$$

Then by taking a special L^2-preserving scaling as:

$$w_c(x)=\varepsilon_c^{\frac{N}{2}}u_c(\varepsilon_c x+\varepsilon_c y_c), \tag{1.13}$$

where

$$\varepsilon_c^2=\frac{2(N+\alpha+2)}{N\int_{\mathbb{R}^N}(I_\alpha*|u_c|^{\frac{N+\alpha+2}{N}})|u_c|^{\frac{N+\alpha+2}{N}}}\to0 \quad \text{as } c\to(c_*)^-$$

and the sequence $\{y_c\}$ is derived from the vanishing lemma, we succeeded in proving that there is a constant $C_2>0$ independent of c such that

$$\int_{\mathbb{R}^N}V(\varepsilon_c x+\varepsilon_c y_c)|w_c(x)|^2\ge C_2\varepsilon_c^q \quad \text{as } c\to(c_*)^-,$$

which and (1.6) implies that $e_c \geq C_3[1-(\frac{c}{c_*})^{\frac{2(\alpha+2)}{N}}]^{\frac{q}{q+2}}$ for some constant $C_3>0$ independent of c. In succession, there exist two constants $0<C_4<C_5$ independent of c such that $C_4[1-(\frac{c}{c_*})^{\frac{2(\alpha+2)}{N}}]^{\frac{-2}{q+2}} \leq \int_{\mathbb{R}^N} |\nabla u_c|^2 \leq C_5[1-(\frac{c}{c_*})^{\frac{2(\alpha+2)}{N}}]^{\frac{-2}{q+2}}$. Finally, by using the Euler-Lagrange equation u_c satisfied and the scaling (1.13) again with $\varepsilon_c = [1-(\frac{c}{c_*})^{\frac{2(\alpha+2)}{N}}]^{\frac{1}{q+2}}$, we show that

$$e_c \approx [1-(\frac{c}{c_*})^{\frac{2(\alpha+2)}{N}}]^{\frac{q}{q+2}} \frac{q+2}{q}\frac{\lambda^2 c_*^2}{2}\left(\frac{N}{\alpha+2}\right)^{\frac{q}{q+2}} \quad \text{as } c \to (c_*)^-,$$

which implies (1.12).

Throughout this paper, we use standard notations. For simplicity, we write $\int_\Omega h$ to mean the Lebesgue integral of $h(x)$ over a domain $\Omega \subset \mathbb{R}^N$. $L^p := L^p(\mathbb{R}^N)$ $(1 \leq p < +\infty)$ is the usual Lebesgue space with the standard norm $|\cdot|_p$. We use " $\to$ " and " $\rightharpoonup$ " to denote the strong and weak convergence in the related function space respectively. C will denote a positive constant unless specified. We use " $:=$ " to denote definitions. We denote a subsequence of a sequence $\{u_n\}$ as $\{u_n\}$ to simplify the notation unless specified.

The paper is organized as follows. In Section 2, we will determine the best constant for the interpolation estimate (1.6) and give the proof of Theorem 1.1. In section 3, we prove Theorems 1.4 and 1.5.

2. Proof of Theorem 1.1

In this section, we first prove the interpolation estimate (1.6). It is enough to consider the following minimization problem:

$$S_p = \inf_{u \in H^1(\mathbb{R}^N)\setminus\{0\}} W_p(u),$$

where

$$W_p(u) = \frac{\left(\int_{\mathbb{R}^N} |\nabla u|^2\right)^{\frac{Np-(N+\alpha)}{2}} \left(\int_{\mathbb{R}^N} |u|^2\right)^{\frac{N+\alpha-(N-2)p}{2}}}{\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p}.$$

Lemma 2.1. *([19], Lemma 2.4) Let $N \geq 1$, $\alpha \in (0,N)$, $p \in [1, \frac{2N}{N+\alpha})$ and $\{u_n\}$ be a bounded sequence in $L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N)$. If $u_n \to u$ a.e. in $\mathbb{R}^N$ as $n \to +\infty$, then*

$$\lim_{n\to+\infty}\left(\int_{\mathbb{R}^N}(I_\alpha * |u_n|^p)|u_n|^p - \int_{\mathbb{R}^N}(I_\alpha * |u_n-u|^p)|u_n-u|^p\right) = \int_{\mathbb{R}^N}(I_\alpha * |u|^p)|u|^p.$$

Lemma 2.2. *([26], Vanishing Lemma) Let $r>0$ and $2 \leq q < 2^*$. If $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$ and*

$$\sup_{y\in\mathbb{R}^N}\int_{B_r(y)} |u_n|^q \to 0, \ n \to +\infty,$$

then $u_n \to 0$ in $L^s(\mathbb{R}^N)$ for $2 < s < 2^*$.

Lemma 2.3. *Let $N \geq 1$, $\alpha \in (0, N)$ and $\frac{N+\alpha}{N} < p < \frac{N+\alpha}{(N-2)_+}$, then S_p is achieved by a function $Q_p \in H^1(\mathbb{R}^N)\backslash\{0\}$, where Q_p is a nontrivial solution of equation* (1.7) *and*

$$S_p = \frac{|Q_p|_2^{2p-2}}{p}.$$

Proof. The lemma can be viewed as a consequence of Proposition 2.1 in [19] and Theorem 9 in [11], but we give an alternative proof here. The idea of the proof comes from [25], but some details are delicate.

Since $W_p(u) \geq 0$ for any $u \in H^1(\mathbb{R}^N)\backslash\{0\}$, S_p is well defined. Let $\{u_n\} \subset H^1(\mathbb{R}^N)\backslash\{0\}$ be a minimizing sequence for S_p, i.e. $W_p(u_n) \to S_p$ as $n \to +\infty$. Set

$$\lambda_n := \frac{(\int_{\mathbb{R}^N} |u_n|^2)^{\frac{N-2}{4}}}{(\int_{\mathbb{R}^N} |\nabla u_n|^2)^{\frac{N}{4}}}, \qquad \mu_n := \frac{(\int_{\mathbb{R}^N} |u_n|^2)^{\frac{1}{2}}}{(\int_{\mathbb{R}^N} |\nabla u_n|^2)^{\frac{1}{2}}}$$

and

$$v_n(x) := \lambda_n u_n(\mu_n x).$$

Then $\int_{\mathbb{R}^N} |v_n|^2 = \int_{\mathbb{R}^N} |\nabla v_n|^2 = 1$ and

$$W_p(v_n) = W_p(u_n) \to S_p \quad \text{as} \quad n \to +\infty, \tag{2.1}$$

i.e. $\{v_n\}$ is a bounded minimizing sequence for S_p.

Let $\delta := \lim_{n\to+\infty} \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |v_n|^2$. If $\delta = 0$, then by Lemma 2.2, $v_n \to 0$ in $L^s(\mathbb{R}^N)$, $2 < s < 2^*$. Hence by the Hardy-Littlewood-Sobolev inequality (1.4),

$$W_p(v_n) = \frac{1}{\int_{\mathbb{R}^N} (I_\alpha * |v_n|^p)|v_n|^p} \to +\infty,$$

which contradicts (2.1). Therefore, $\delta > 0$ and there exists a sequence $\{y_n\} \subset \mathbb{R}^N$ such that

$$\int_{B_1(y_n)} |v_n|^2 \geq \frac{\delta}{2} > 0. \tag{2.2}$$

Up to translations, we may assume that $y_n = 0$. Since $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$ and by (2.2), there exists $v_p \in H^1(\mathbb{R}^N)\backslash\{0\}$ such that $v_n \rightharpoonup v_p$ in $H^1(\mathbb{R}^N)$. Then by the Brezis Lemma and Lemma 2.1, we have

$$\begin{aligned}
S_p &\leq W_p(v_p) \\
&\leq \lim_{n\to+\infty} \left[W_p(v_n) \frac{\int_{\mathbb{R}^N} (I_\alpha * |v_n|^p)|v_n|^p}{\int_{\mathbb{R}^N} (I_\alpha * |v_p|^p)|v_p|^p} - W_p(v_n - v_p) \frac{\int_{\mathbb{R}^N} (I_\alpha * |v_n - v_p|^p)|v_n - v_p|^p}{\int_{\mathbb{R}^N} (I_\alpha * |v_p|^p)|v_p|^p} \right] \\
&\leq S_p \lim_{n\to+\infty} \left(\frac{\int_{\mathbb{R}^N} (I_\alpha * |v_n|^p)|v_n|^p - \int_{\mathbb{R}^N} (I_\alpha * |v_n - v_p|^p)|v_n - v_p|^p}{\int_{\mathbb{R}^N} (I_\alpha * |v_p|^p)|v_p|^p} \right) \\
&= S_p,
\end{aligned}$$

i.e. $W_p(v_p) = S_p$. Moreover, $|\nabla v_p|_2 = |v_p|_2 = 1$ and $S_p = \frac{1}{\int_{\mathbb{R}^N} (I_\alpha * |v_p|^p)|v_p|^p}$.

Therefore, for any $h \in H^1(\mathbb{R}^N)$, $\frac{d}{dt}\Big|_{t=0} W_p(v_p + th) = 0$, i.e. v_p satisfies the following equation

$$-[Np-(N+\alpha)]\Delta v_p + [N+\alpha-(N-2)p]v_p = 2pS_p(I_\alpha * |v|^p)|v_p|^{p-2}v_p, \text{ in } \mathbb{R}^N.$$

Let $v_p = (\frac{1}{pS_p})^{\frac{1}{2p-2}} Q_p$, then Q_p is a nontrivial solution of (1.7) and $S_p = \frac{|Q_p|_2^{2p-2}}{p}$.

□

Next we give the proof of Theorem 1.1. For any $u \in S(c)$, set

$$A(u) := \int_{\mathbb{R}^N} |\nabla u|^2, \qquad B(u) := \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p,$$

then $I_p(u) = \frac{1}{2}A(u) - \frac{1}{2p}B(u)$. It follows from (1.6)(1.7) that for $\frac{N+\alpha}{N} < p < \frac{N+\alpha}{(N-2)_+}$,

$$B(u) \le \frac{p}{|Q_p|_2^{2p-2}} A(u)^{\frac{Np-(N+\alpha)}{2}} c^{N+\alpha-(N-2)p} \tag{2.3}$$

with equality for $u = Q_p$ given in (1.7), moreover,

$$A(Q_p) = \frac{1}{p}B(Q_p) = |Q_p|_2^2. \tag{2.4}$$

Lemma 2.4. *Let $N \ge 1$ and $\alpha \in (0, N)$.*

(1) *If $\frac{N+\alpha}{N} < p < \frac{N+\alpha+2}{N}$, then $I_p(u)$ is bounded from below and coercive on $S(c)$ for all $c > 0$, moreover, $I_p(c^2) < 0$.*

(2) *If $p = \frac{N+\alpha+2}{N}$, then* $I_{\frac{N+\alpha+2}{N}}(c^2) = \begin{cases} 0, & 0 < c \le c_* := |Q_{\frac{N+\alpha+2}{N}}|_2, \\ -\infty, & c > c_*, \end{cases}$

(3) *If $\frac{N+\alpha+2}{N} < p < \frac{N+\alpha}{(N-2)_+}$, then $I_p(c^2) = -\infty$ for all $c > 0$.*

Proof. (1) For any $c > 0$ and $u \in S(c)$, by (2.3), there exists $C := \frac{c^{N+\alpha-(N-2)p}}{|Q_p|_2^{2p-2}}$ such that

$$I_p(u) \ge \frac{A(u) - CA(u)^{\frac{Np-(N+\alpha)}{2}}}{2}. \tag{2.5}$$

Since $\frac{N+\alpha}{N} < p < \frac{N+\alpha+2}{N}$, $0 < Np - (N+\alpha) < 2$. Then (2.5) implies that $I_p(u)$ is bounded from below and coercive on $S(c)$ for any $c > 0$.

Set $u^t(x) := t^{\frac{N}{2}} u(tx)$ with $t > 0$, then $u^t \in S(c)$ and

$$I_p(u^t) = \frac{t^2}{2}A(u) - \frac{t^{Np-(N+\alpha)}}{2p}B(u) < 0 \qquad \text{for } t > 0 \text{ small enough} \tag{2.6}$$

since $0 < Np - (N+\alpha) < 2$, which implies that $I_p(c^2) < 0$ for each $c > 0$.

(2) When $p = \frac{N+\alpha+2}{N}$, $Np - (N+\alpha) = 2$, similarly to (2.5) and (2.6), we have

$$I_{\frac{N+\alpha+2}{N}}(u) \ge \frac{A(u)}{2}\left[1 - \left(\frac{c}{c_*}\right)^{\frac{2(\alpha+2)}{N}}\right] \ge 0 \quad \text{if} \quad 0 < c \le c_*$$

and $I_{\frac{N+\alpha+2}{N}}(c^2) \le I_{\frac{N+\alpha+2}{N}}(u^t) \to 0$ as $t \to 0^+$ for all c. Then $I_{\frac{N+\alpha+2}{N}}(c^2) = 0$ if $0 < c \le c_*$.

If $c > c_*$, set $Q^t(x) := \frac{ct^{\frac{N}{2}}}{c_*} Q_{\frac{N+\alpha+2}{N}}(tx)$, then by (2.4),

$$I_{\frac{N+\alpha+2}{N}}(Q^t) = \frac{c^2t^2}{2c_*^2}\left[1 - \left(\frac{c}{c_*}\right)^{\frac{2(\alpha+2)}{N}}\right] \to -\infty \quad \text{as} \quad t \to +\infty,$$

then $I_{\frac{N+\alpha+2}{N}}(c^2) = -\infty$ for $c > c_*$.

(3) If $\frac{N+\alpha+2}{N} < p < \frac{N+\alpha}{(N-2)_+}$, then $Np-(N+\alpha) > 2$, hence by (2.6), we have $I_p(u^t) \to -\infty$ as $t \to +\infty$, so $I_p(c^2) = -\infty$ for all $c > 0$. □

Lemma 2.5. *If $\frac{N+\alpha}{N} < p < \frac{N+\alpha+2}{N}$, then*

(1) *the function $c \mapsto I_p(c^2)$ is continuous on $(0, +\infty)$;*

(2)
$$I_p(c^2) < I_p(\alpha^2) + I_p(c^2 - \alpha^2), \qquad \forall\ 0 < \alpha < c < +\infty. \tag{2.7}$$

Proof. The proof of (1) follows from Lemma 2.4 and is similar to that of Theorem 2.1 in [2], so we omit it.

(2) For any $c > 0$, let $\{u_n\} \subset S(c)$ be a minimizing sequence for $I_p(c^2) < 0$, then by Lemma 2.4, $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$ and there exists a constant $K_1 > 0$ independent of n such that $B(u_n) \geq K_1$. Set $u_n^\theta = \theta u_n$ with $\theta > 1$, then $u_n^\theta \in S(\theta c)$ and

$$I_p(u_n^\theta) - \theta^2 I(u_n) = \frac{\theta^2 - \theta^{2p}}{2p} B(u_n) \leq \frac{\theta^2 - \theta^{2p}}{2p} K_1 < 0.$$

Letting $n \to +\infty$, we have $I_p(\theta^2 c^2) < I_p(c^2)$, $\theta > 1$, which easily implies (2.7) by using Lemma 2.4 (1). □

Lemma 2.6. *Let $N \geq 1$, $\alpha \in (0, N)$ and $\frac{N+\alpha}{N} < p < \frac{N+\alpha}{(N-2)_+}$. If u is a critical point of $I_p(u)$ constrained on $S(c)$, then there exists $\mu_c < 0$ such that $I_p'(u) - \mu_c u = 0$ in $H^{-1}(\mathbb{R}^N)$ and*

$$A(u) - \frac{Np - (N+\alpha)}{2p} B(u) = 0.$$

Proof. Since $(I_p|_{S(c)})'(u) = 0$, there exists $\mu_c \in \mathbb{R}$ such that $I_p'(u) - \mu_c u = 0$ in $H^{-1}(\mathbb{R}^N)$. Then

$$A(u) - B(u) = \mu_c c^2.$$

By Proposition 3.5 in [20], u satisfies the following Pohozaev identity,

$$\frac{N-2}{2} A(u) - \frac{N+\alpha}{2p} B(u) = \frac{N}{2} \mu_c c^2.$$

Hence $A(u) = \frac{Np-(N+\alpha)}{2p} B(u)$ and

$$\mu_c = \frac{(N-2)p - (N+\alpha)}{2pc^2} B(u) < 0.$$

□

Proof of Theorem 1.1

Proof. (1) If $p = \frac{N+\alpha}{N}$, for any $c > 0$ and $u \in S(c)$, by (1.8) we have

$$I_{\frac{N+\alpha}{N}}(u) \geq -\frac{N}{2(N+\alpha)} \left(\frac{c}{|Q_{\frac{N+\alpha}{N}}|_2} \right)^{\frac{2(N+\alpha)}{N}}.$$

Set $Q^t_{\frac{N+\alpha}{N}}(x) := \frac{ct^{\frac{N}{2}}}{|Q_{\frac{N+\alpha}{N}}|_2} Q_{\frac{N+\alpha}{N}}(tx)$, then by (1.8) again, we see that

$$I_{\frac{N+\alpha}{N}}(Q^t_{\frac{N+\alpha}{N}}) = \frac{c^2 t^2}{2|Q_{\frac{N+\alpha}{N}}|_2^2} A(Q_{\frac{N+\alpha}{N}}) - \frac{N}{2(N+\alpha)} \left(\frac{c}{|Q_{\frac{N+\alpha}{N}}|_2} \right)^{\frac{2(N+\alpha)}{N}},$$

letting $t \to 0^+$, then $I_{\frac{N+\alpha}{N}}(c^2) = -\frac{N}{2(N+\alpha)} (\frac{c}{|Q_{\frac{N+\alpha}{N}}|_2})^{\frac{2(N+\alpha)}{N}}$.

By contradiction, if for some $c > 0$, there is $u \in S(c)$ such that $I_{\frac{N+\alpha}{N}}(u) = I_{\frac{N+\alpha}{N}}(c^2)$, then (1.8) shows that

$$0 \leq \frac{1}{2} A(u) = \frac{N}{2(N+\alpha)} \left[B(u) - \left(\frac{c}{|Q_{\frac{N+\alpha}{N}}|_2} \right)^{\frac{2(N+\alpha)}{N}} \right] \leq 0,$$

which implies that $u = 0$. It is a contradiction. So $I_{\frac{N+\alpha}{N}}(c^2)$ has no minimizer for all $c > 0$.

(2) If $\frac{N+\alpha}{N} < p < \frac{N+\alpha+2}{N}$, for any $c > 0$, by Lemma 2.4, $I_p(c^2) < 0$. Let $\{u_n\} \subset S(c)$ be a minimizing sequence for $I_p(c^2)$, then Lemma 2.4 (1) implies that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$ and for some constant $C > 0$ independent of n, $B(u_n) \geq C$. Hence there exists $u \in H^1(\mathbb{R}^N)$ such that

$$u_n \rightharpoonup u \text{ in } H^1(\mathbb{R}^N), \qquad u_n(x) \to u(x) \text{ a.e. in } \mathbb{R}^N. \tag{2.8}$$

Moreover, by the Vanishing Lemma 2.2, up to translations, we may assume that $u \neq 0$. Then $0 < |u|_2 := \alpha \leq c$. We just suppose that $\alpha < c$, then $u \in S(\alpha)$. By (2.8) and the Brezis lemma, we have

$$\lim_{n \to +\infty} |u_n - u|_2^2 = \lim_{n \to +\infty} |u_n|_2^2 - |u|_2^2 = c^2 - \alpha^2.$$

Then by Lemma 2.1 and Lemma 2.5 (1), we have

$$I_p(c^2) = \lim_{n \to +\infty} I_p(u_n) = \lim_{n \to +\infty} I_p(u_n - u) + I_p(u) \geq I_p(c^2 - \alpha^2) + I_p(\alpha^2),$$

which contradicts (2.7). So $|u|_2 = c$, i.e. $u_n \to u$ in $L^2(\mathbb{R}^N)$. By (2.3), we have $B(u_n) \to B(u)$. Then

$$I_p(c^2) \leq I_p(u) \leq \lim_{n \to +\infty} I_p(u_n) = I_p(c^2),$$

i.e. u is minimizer for $I_p(c^2)$.

(3) (i) has been proved in Lemma 2.5 (2). To prove (ii), by contradiction, if there exists $c_0 \in (0, c_*)$ such that $I_{\frac{N+\alpha+2}{N}}(c_0^2)$ has a minimizer $u_0 \in S(c_0)$, i.e. $I_{\frac{N+\alpha+2}{N}}(u_0) = I_{\frac{N+\alpha+2}{N}}(c_0^2) =$

0, then by (2.3),

$$A(u_0)=\frac{N}{N+\alpha+2}B(u_0)\le\left(\frac{c_0}{c_*}\right)^{\frac{2(\alpha+2)}{N}}A(u_0)<A(u_0),$$

which is impossible. So combining (i), we see that $I_{\frac{N+\alpha+2}{N}}(c^2)$ has no minimizer for all $c\neq c_*$.

By (2.4), we see that $I_{\frac{N+\alpha+2}{N}}(Q_{\frac{N+\alpha+2}{N}})=0=I_{\frac{N+\alpha+2}{N}}(c_*^2)$, i.e. $Q_{\frac{N+\alpha+2}{N}}$ is a minimizer for $I_{\frac{N+\alpha+2}{N}}(c_*^2)$. Moreover, by Lemmas 3.1 (2) and 3.2 below, each groundstate solution of (1.7) is a minimizer of $I_{\frac{N+\alpha+2}{N}}(c_*^2)$. So we proved (iii).

For any $c>0$, suppose that u is a critical point of $I_{\frac{N+\alpha+2}{N}}(u)$ constrained on $S(c)$, then by (2.4) and Lemma 2.6, we have

$$A(u)=\frac{N}{N+\alpha+2}B(u)\le\left(\frac{c}{c_*}\right)^{\frac{2(\alpha+2)}{N}}A(u),$$

which implies that $c_*\le c$. Therefore, there exists no critical point for $I_{\frac{N+\alpha+2}{N}}(u)$ constrained on $S(c)$ if $0<c<c_*$. So (iv) is proved.

(4) By Lemma 2.4 (3), $I_p(c^2)$ has no minimizer for all $c>0$ if $\frac{N+\alpha+2}{N}<p<\frac{N+\alpha}{(N-2)_+}$. □

3. Proof of Theorems 1.4 and 1.5

For $p=\frac{N+\alpha+2}{N}$, (2.3) turns to be

$$B(u)\le\frac{N+\alpha+2}{N}\left(\frac{1}{c_*}\right)^{\frac{2(\alpha+2)}{N}}A(u)|u|_2^{\frac{2(\alpha+2)}{N}},\tag{3.1}$$

with equality for $u=Q_{\frac{N+\alpha+2}{N}}$ and $c_*:=|Q_{\frac{N+\alpha+2}{N}}|_2$, where $Q_{\frac{N+\alpha+2}{N}}$ is a nontrivial solution of

$$-\Delta Q_{\frac{N+\alpha+2}{N}}+\frac{\alpha+2}{N}Q_{\frac{N+\alpha+2}{N}}=(I_\alpha*|Q_{\frac{N+\alpha+2}{N}}|^{\frac{N+\alpha+2}{N}})|Q_{\frac{N+\alpha+2}{N}}|^{\frac{N+\alpha+2}{N}-2}Q_{\frac{N+\alpha+2}{N}},\quad\text{in }\mathbb{R}^N.$$

Set $Q_{\frac{N+\alpha+2}{N}}(x)=\left(\sqrt{\frac{\alpha+2}{N}}\right)^{\frac{N}{2}}\widetilde{Q}_{\frac{N+\alpha+2}{N}}(\sqrt{\frac{\alpha+2}{N}}x)$, then $\widetilde{Q}_{\frac{N+\alpha+2}{N}}$ satisfies the equation

$$-\Delta\widetilde{Q}_{\frac{N+\alpha+2}{N}}+\widetilde{Q}_{\frac{N+\alpha+2}{N}}=(I_\alpha*|\widetilde{Q}_{\frac{N+\alpha+2}{N}}|^{\frac{N+\alpha+2}{N}})|\widetilde{Q}_{\frac{N+\alpha+2}{N}}|^{\frac{N+\alpha+2}{N}-2}\widetilde{Q}_{\frac{N+\alpha+2}{N}},\quad\text{in }\mathbb{R}^N.\tag{3.2}$$

The following Lemma is a direct conclusion of Theorems 1-4 in [19].

Lemma 3.1. *Assume that $N\ge1$ and $\alpha\in(0,N)$.*

(1) *There is at least one groundstate solution $u\in H^1(\mathbb{R}^N)$ to (3.2) with*

$$F(u)=d:=\inf\{F(v)|\ v\in H^1(\mathbb{R}^N)\backslash\{0\}\ \textit{is a weak solution of}\ (3.2)\},$$

where $F(v)=\dfrac{1}{2}\displaystyle\int_{\mathbb{R}^N}(|\nabla v|^2+|v|^2)-\frac{N}{2(N+\alpha+2)}\int_{\mathbb{R}^N}(I_\alpha*|v|^{\frac{N+\alpha+2}{N}})|v|^{\frac{N+\alpha+2}{N}}.$

(2) *If $u \in H^1(\mathbb{R}^N)$ is a nontrivial solution of* (3.2), *then $u \in L^1(\mathbb{R}^N) \cap C^2(\mathbb{R}^N)$, $u \in W^{2,s}(\mathbb{R}^N)$ for every $s > 1$ and $u \in C^\infty(\mathbb{R}^N \backslash u^{-1}(\{0\})$. Moreover,*

$$\frac{N+\alpha+2}{N} A(u) = \frac{N+\alpha+2}{\alpha+2} \int_{\mathbb{R}^N} |u|^2 = B(u). \tag{3.3}$$

(3) *If u is a groundstate solution of* (3.2), *then u is either positive or negative and there exists $x_0 \in \mathbb{R}^N$ and a monotone function $v \in C^\infty(0, +\infty)$ such that*

$$u(x) = v(|x - x_0|),\ \forall\ x \in \mathbb{R}^N.$$

(4) *Let $N - 2 \leq \alpha < N$ if $N \geq 3$ and $0 < \alpha < N$ if $N = 1, 2$. If u is a groundstate solution of* (3.2), *then*

$$\lim_{|x| \to +\infty} |u(x)||x|^{\frac{N-1}{2}} e^{|x|} \in (0, +\infty).$$

Moreover, $|\nabla u(x)| = O(|x|^{-\frac{N-1}{2}} e^{-|x|})$ as $|x| \to +\infty$.

Lemma 3.2. *(1) $d = \frac{c_*^2}{2}$.*

(2) u is a nontrivial solution of (3.2) *with $|u|_2 = c_*$ if and only if u is a groundstate solution.*

Proof. For any nontrivial solution u of (3.2), then by Lemma 3.1 (1)(2) and (3.1), we have

$$c_* \leq |u|_2$$

and

$$d \leq F(u) = \frac{1}{2} \int_{\mathbb{R}^N} |u|^2$$

where equality holds only if u is a groundstate solution. In particular, since $\widetilde{Q}_{\frac{N+\alpha+2}{N}}$ is a nontrivial solution of (3.2),

$$d \leq F(\widetilde{Q}_{\frac{N+\alpha+2}{N}}) = \frac{|\widetilde{Q}_{\frac{N+\alpha+2}{N}}|_2^2}{2} = \frac{c_*^2}{2}.$$

Therefore, if u is a groundstate solution of (3.2), then by Lemma 3.1 (3), u is nontrivial and

$$\frac{c_*^2}{2} \leq \frac{|u|_2^2}{2} = F(u) = d \leq \frac{c_*^2}{2},$$

which shows that $d = \frac{c_*^2}{2}$ and $|u|_2 = c_*$.

On the other hand, if u is a nontrivial solution of (3.2) with $|u|_2 = c_*$, then

$$\frac{c_*^2}{2} = d \leq F(u) = \frac{1}{2} \int_{\mathbb{R}^N} |u|^2 = \frac{c_*^2}{2},$$

which implies that $F(u) = d$, i.e. u is a groundstate solution. □

Remark 3.3. $\widetilde{Q}_{\frac{N+\alpha+2}{N}}$ is a groundstate solution of (3.2).

Lemma 3.4. *([1]) Suppose that $V \in L^\infty_{loc}(\mathbb{R}^N)$ and $\lim_{|x| \to +\infty} V(x) = +\infty$, then the embedding $\mathcal{H} \hookrightarrow L^s(\mathbb{R}^N)$, $2 \leq s < 2^*$ is compact.*

Proof of Theorem 1.4

Proof. Set

$$C(u) := \int_{\mathbb{R}^N} V(x)|u|^2 \geq 0, \quad \forall\, u \in H^1(\mathbb{R}^N),$$

then

$$E(u) = \frac{A(u)}{2} + \frac{C(u)}{2} - \frac{N}{2(N+\alpha+2)} B(u).$$

(1) By (3.1), for any $0 < c \leq c_*$ and $u \in \widetilde{S}(c)$,

$$E(u) \geq \frac{1}{2}\left[1 - \left(\frac{c}{c_*}\right)^{\frac{2(\alpha+2)}{N}}\right] A(u) + \frac{1}{2} C(u) \geq 0, \tag{3.4}$$

then $e_c = \inf\limits_{u \in \widetilde{S}(c)} E(u) \geq 0$ is well defined for $0 < c \leq c_*$.

For each $0 < c < c_*$, let $\{u_n\} \subset \widetilde{S}(c)$ be a minimizing sequence for e_c, then by (3.4), $\{u_n\}$ is bounded in $\mathcal{H}$. Hence there exists $u_c \in \mathcal{H}$ such that $u_n \rightharpoonup u_c$ in $\mathcal{H}$. By Lemma 3.4, $u_n \to u_c$ in $L^s(\mathbb{R}^N)$, $2 \leq s < 2^*$, which implies that $|u_c|_2 = c$ and $B(u_n) \to B(u_c)$. So $e_c \leq E(u_c) \leq \lim\limits_{n\to+\infty} E(u_n) = e_c$, i.e. $u_c \in \widetilde{S}(c)$ is a minimizer of e_c. Moreover, by (3.4), $e_c > 0$. So $e_c > 0$ has at least one minimizer for all $0 < c < c_*$.

(2) Let $N-2 \leq \alpha < N$ if $N \geq 3$ and $0 < \alpha < N$ if $N = 1, 2$. For any $c > 0$, let $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that $0 \leq \varphi(x) \leq 1$, $\varphi(x) \equiv 1$ for $|x| \leq 1$, $\varphi(x) \equiv 0$ for $|x| \geq 2$ and $|\nabla\varphi| \leq 2$. For any $x_0 \in \mathbb{R}^N$ and any $t > 0$, set

$$\widetilde{Q}^t(x) = \frac{cA_t t^{\frac{N}{2}}}{c_*}\varphi(x-x_0)\widetilde{Q}_{\frac{N+\alpha+2}{N}}(t(x-x_0)), \tag{3.5}$$

where $A_t > 0$ is chosen to satisfy that $|\widetilde{Q}^t|_2 = c$. By the exponential decay of $\widetilde{Q}_{\frac{N+\alpha+2}{N}}$, we see that

$$\frac{1}{A_t^2} = 1 + \frac{1}{c_*^2}\int_{\mathbb{R}^N}\left(\varphi^2(\frac{x}{t}) - 1\right)|\widetilde{Q}_{\frac{N+\alpha+2}{N}}(x)|^2 \to 1$$

as $t \to +\infty$. Then A_t depends only on t and $\lim\limits_{t\to+\infty} A_t = 1$. Since $V(x)\varphi^2(x-x_0)$ is bounded and has compact support, $C(\widetilde{Q}^t) \to \frac{c^2}{c_*^2}V(x_0)$.

$$\begin{aligned}
B(\widetilde{Q}^t) &= (\frac{cA_t}{c_*})^{\frac{2(N+\alpha+2)}{N}} t^2 \Big\{ B(\widetilde{Q}_{\frac{N+\alpha+2}{N}}) \\
&\quad + \int_{\mathbb{R}^N} \{I_\alpha * [(|\varphi(\frac{y}{t})|^{\frac{N+\alpha+2}{N}} - 1)|\widetilde{Q}_{\frac{N+\alpha+2}{N}}(y)|^{\frac{N+\alpha+2}{N}}]\}(|\varphi(\frac{x}{t})|^{\frac{N+\alpha+2}{N}} + 1)|\widetilde{Q}_{\frac{N+\alpha+2}{N}}(x)|^{\frac{N+\alpha+2}{N}} \Big\} \\
&:= (\frac{cA_t}{c_*})^{\frac{2(N+\alpha+2)}{N}} t^2 \left[B(\widetilde{Q}_{\frac{N+\alpha+2}{N}}) + f_1(t) \right].
\end{aligned}$$

By the Hardy-Littlewood-Sobolev inequality (1.4) and the exponential decay of $\widetilde{Q}_{\frac{N+\alpha+2}{N}}$, we have there exists a constant $C>0$ such that

$$\begin{aligned}|f_1(t)| &\leq C\left(\int_{\mathbb{R}^N}|[\varphi(\frac{x}{t})]^{\frac{N+\alpha+2}{N}}-1|^{\frac{2N}{N+\alpha}}|\widetilde{Q}_{\frac{N+\alpha+2}{N}}(x)|^{\frac{2(N+\alpha+2)}{N+\alpha}}\right)^{\frac{N+\alpha}{2N}}\\ &\leq C\left(\int_{|x|\geq t}|\widetilde{Q}_{\frac{N+\alpha+2}{N}}(x)|^{\frac{2(N+\alpha+2)}{N+\alpha}}\right)^{\frac{N+\alpha}{2N}}\\ &\leq C\left(\int_t^{+\infty}r^{-\frac{2(N-1)}{N+\alpha}}e^{-\frac{2(N+\alpha+2)}{N+\alpha}r}\right)^{\frac{N+\alpha}{2N}}\leq Ct^{-\frac{2(N-1)}{2N}}e^{-\frac{N+\alpha+2}{N}t} \qquad \text{as } \ t\to+\infty.\end{aligned}$$

Then by the exponential decay of $\widetilde{Q}_{\frac{N+\alpha+2}{N}}$ and $|\nabla\widetilde{Q}_{\frac{N+\alpha+2}{N}}|$, we have

$$E(\widetilde{Q}^t)=\frac{c^2}{2c_*^2}t^2A(\widetilde{Q}_{\frac{N+\alpha+2}{N}})\left[1-\left(\frac{c}{c_*}\right)^{\frac{2(\alpha+2)}{N}}\right]+t^2f_2(t)+\frac{c^2}{2c_*^2}V(x_0) \ \text{ as } \ t\to+\infty, \quad (3.6)$$

where $f_2(t)$ denotes a function satisfying that $\lim\limits_{t\to+\infty}|f_2(t)|t^r=0$ for all $r>0$.

If $c>c^*$, then by (3.6), $e_c\leq\lim\limits_{t\to+\infty}E(\widetilde{Q}^t)=-\infty$, hence $e_c=-\infty$ and there exists no minimizer for e_c.

If $c=c^*$, then by (3.4) and (3.6), $0\leq e_{c_*}\leq\frac{V(x_0)}{2}$. Taking the infimum over x_0, $e_{c_*}=0$. We just suppose that there exists $u\in\widetilde{S}(c_*)$ such that $E(u)=e_{c_*}$, then it follows from (3.4) that

$$C(u)=0, \quad (3.7)$$

which and the condition (V_0) imply that u must have compact support. On the other hand, $(E|_{\widetilde{S}(c_*)})'(u)=0$. Then there exists $\mu_{c_*}\in\mathbb{R}$ such that $E'(u)-\mu_{c_*}u=0$, i.e. for any $h\in C_0^\infty(\mathbb{R}^N)$,

$$\begin{aligned}0 &= \langle E'(u)-\mu_{c_*}u,h\rangle\\ &=\int_{\mathbb{R}^N}(\nabla u\nabla h-\mu_{c_*}uh)-\int_{\mathbb{R}^N}(I_\alpha*|u|^{\frac{N+\alpha+2}{N}})|u|^{\frac{N+\alpha+2}{N}-2}uh \quad (3.8)\\ &=\langle I'_{\frac{N+\alpha+2}{N}}(u)-\mu_{c_*}u,h\rangle,\end{aligned}$$

where we have used the fact that $\int_{\mathbb{R}^N}V(x)uh=0$ due to the Hölder inequality and (3.7). Then by Lemma 2.6, we see that $\mu_{c_*}<0$. Set $u(x):=(\sqrt{-\mu_{c_*}})^{\frac{N}{2}}w(\sqrt{-\mu_{c_*}}x)$, then by (3.8), w is a nontrivial solution of (3.2) with $|w|_2=c_*$, hence by Lemma 3.2 w is a groundstate solution. So by Lemma 3.1 (4), $\lim\limits_{|x|\to+\infty}|u(x)||x|^{\frac{N-1}{2}}e^{|x|}\in(0,+\infty)$, which contradicts (3.7).

Moreover, we conclude from (3.5) and (3.6) that $\limsup\limits_{c\to(c_*)^-}e_c\leq\frac{V(x_0)}{2}$ as $t\to+\infty$. By the arbitrary of x_0, we have $\lim\limits_{c\to(c_*)^-}e_c=0=e_{c_*}$. $\square$

In the following, we consider the concentration behavior of minimizers as c approaches c_* from below when $N-2 \leq \alpha < N$ if $N \geq 3$ and $0 < \alpha < N$ if $N = 1, 2$ and the potential $V(x)$ satisfies conditions $(V_0)(V_1)$.

Lemma 3.5. *Suppose that $(V_0)(V_1)$ hold, then there exist two positive constants $M_1 < M_2$ independent of c such that*

$$M_1 \left[1 - \left(\frac{c}{c_*}\right)^{\frac{2(\alpha+2)}{N}}\right]^{\frac{q}{q+2}} \leq e_c \leq M_2 \left[1 - \left(\frac{c}{c_*}\right)^{\frac{2(\alpha+2)}{N}}\right]^{\frac{q}{q+2}} \qquad as \quad c \to (c_*)^-,$$

where $q = \max\{q_1, q_2, \cdots, q_m\}$.

Proof. The proof consists of two steps.

Step 1. Without loss of generality, we may assume that $q = q_{i_0}$ for some $1 \leq i_0 \leq m$. By (V_1), there exists $R > 0$ small such that $V(x) \leq 2\mu_{i_0}|x - x_{i_0}|^{q_{i_0}}$ for $|x - x_{i_0}| \leq R$. Similarly to (3.5), let

$$u(x) := \frac{cA_{R,t}t^{\frac{N}{2}}}{c_*}\varphi\left(\frac{2(x - x_{i_0})}{R}\right)\widetilde{Q}_{\frac{N+\alpha+2}{N}}(t(x - x_{i_0})) \in \widetilde{S}(c),$$

where $A_{R,t} > 0$ and $A_{R,t} \to 1$ as $t \to +\infty$. Then

$$C(u) \leq \frac{2\mu_{i_0}c^2A_{R,t}^2}{c_*^2}t^{-q_{i_0}}\int_{\mathbb{R}^N}|x|^{q_{i_0}}|\widetilde{Q}_{\frac{N+\alpha+2}{N}}|^2.$$

Hence similarly to (3.6), for large t,

$$e_c \leq \frac{A(\widetilde{Q}_{\frac{N+\alpha+2}{N}})}{2}t^2\left[1 - \left(\frac{c}{c_*}\right)^{\frac{2(\alpha+2)}{N}}\right] + 2\mu_{i_0}t^{-q_{i_0}}\int_{\mathbb{R}^N}|x|^{q_{i_0}}|\widetilde{Q}_{\frac{N+\alpha+2}{N}}(x)|^2 + t^2h(t),$$

where $\lim\limits_{t\to+\infty}|h(t)|t^2 = 0$. By taking $t = [1 - (\frac{c}{c_*})^{\frac{2(\alpha+2)}{N}}]^{-\frac{1}{q_{i_0}+2}}$, then there exists a constant $M_2 > 0$ independent of c such that

$$e_c \leq M_2\left[1 - \left(\frac{c}{c_*}\right)^{\frac{2(\alpha+2)}{N}}\right]^{\frac{q}{q+2}}.$$

Step 2. For any $0 < c < c_*$, there exists $u_c \in \widetilde{S}(c)$ such that $E(u_c) = e_c$. By (3.4) and Theorem 1.4, we see that

$$C(u_c) \leq e_c \to 0 \quad \text{as} \quad c \to (c_*)^-. \tag{3.9}$$

We claim that

$$A(u_c) \to +\infty \quad \text{as} \quad c \to (c_*)^-. \tag{3.10}$$

In fact, by contradiction, if there exists a sequence $\{c_k\} \subset (0, c_*)$ with $c_k \to c_*$ as $k \to +\infty$ such that the sequence of minimizers $\{u_{c_k}\} \subset \widetilde{S}(c_k)$ is uniformly bounded in $\mathcal{H}$, then we may assume that for some $u \in \mathcal{H}$, $u_{c_k} \rightharpoonup u$ in $\mathcal{H}$ and by Lemma 3.4 and (3.1),

$$u_{c_k} \to u \quad \text{in } L^2(\mathbb{R}^N) \quad \text{and} \quad B(u_{c_k}) \to B(u).$$

Hence $u \in \widetilde{S}(c_*)$ and $0 \le e_{c_*} \le E(u) \le \lim_{k\to+\infty} E(u_{c_k}) = \lim_{k\to+\infty} e_{c_k} = 0$, i.e. u is a minimizer of e_{c_*}, which contradicts Theorem 1.4.

Since
$$0 \le \frac{1}{2}A(u_c) - \frac{N}{2(N+\alpha+2)}B(u_c) \le e_c,$$
we see that
$$\lim_{c\to(c_*)^-} \frac{\frac{N}{N+\alpha+2}B(u_c)}{A(u_c)} = 1.$$
Then by (3.10), set
$$\varepsilon_c^{-2} := \frac{N}{2(N+\alpha+2)}B(u_c) \to +\infty \quad \text{as} \quad c \to (c_*)^- \tag{3.11}$$
and $\tilde{w}_c(x) := \varepsilon_c^{\frac{N}{2}} u_c(\varepsilon_c x)$. Then $|\tilde{w}_c|_2 = c$ and
$$\frac{N}{2(N+\alpha+2)}B(\tilde{w}_c) = 1, \qquad 2 \le A(\tilde{w}_c) \le 2 + 2\varepsilon_c^2 e_c. \tag{3.12}$$
Let $\delta := \lim_{c\to(c_*)^-} \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |\tilde{w}_c|^2$. If $\delta = 0$, then $\tilde{w}_c \to 0$ in $L^s(\mathbb{R}^N)$ as $c \to (c_*)^-$, $2 < s < 2^*$, hence by (1.4), $B(\tilde{w}_c) \to 0$, which contradicts (3.12). So $\delta > 0$ and there exists $\{y_c\} \subset \mathbb{R}^N$ such that $\int_{B_1(y_c)} |\tilde{w}_c|^2 \ge \frac{\delta}{2} > 0$. Set
$$w_c(x) := \tilde{w}_c(x + y_c) = \varepsilon_c^{\frac{N}{2}} u_c(\varepsilon_c x + \varepsilon_c y_c),$$
then
$$\int_{B_1(0)} |w_c|^2 \ge \frac{\delta}{2} > 0. \tag{3.13}$$

We claim that $\{\varepsilon_c y_c\}$ is uniformly bounded as $c \to (c_*)^-$. Indeed, if there exists a sequence $\{c_k\} \subset (0, c_*)$ with $c_k \to c_*$ as $k \to +\infty$ such that $|\varepsilon_{c_k} y_{c_k}| \to +\infty$ as $k \to +\infty$, then by (V_0), (3.9) and (3.13) and the Fatou's Lemma, we have
$$\begin{aligned}
0 = \liminf_{k\to+\infty} \int_{\mathbb{R}^N} V(x)|u_{c_k}|^2 &= \liminf_{k\to+\infty} \int_{\mathbb{R}^N} V(\varepsilon_{c_k} x + \varepsilon_{c_k} y_{c_k})|w_{c_k}(x)|^2 \\
&\ge \int_{\mathbb{R}^N} \liminf_{k\to+\infty} [V(\varepsilon_{c_k} x + \varepsilon_{c_k} y_{c_k})|w_{c_k}(x)|^2] \\
&\ge \int_{B_1(0)} \liminf_{k\to+\infty} [V(\varepsilon_{c_k} x + \varepsilon_{c_k} y_{c_k})|w_{c_k}(x)|^2] \\
&\ge (+\infty) \cdot \frac{\delta}{2} = +\infty,
\end{aligned}$$
which is impossible. So $\{\varepsilon_c y_c\}$ is uniformly bounded as $c \to (c_*)^-$. Moreover, there exists $x_{j_0} \in \{x_1, \cdots, x_m\}$ such that
$$\left\{ \frac{\varepsilon_c y_c - x_{j_0}}{\varepsilon_c} \right\} \text{ is uniformly bounded as } c \to (c_*)^-. \tag{3.14}$$

Indeed, by contradiction, we just suppose that for any $x_i \in \{x_1, \cdots, x_m\}$, there exists $c_k \to (c_*)^-$ as $k \to +\infty$ such that $|\frac{\varepsilon_{c_k} y_{c_k} - x_i}{\varepsilon_{c_k}}| \to +\infty$ as $k \to +\infty$. By (V_1), (3.13) and the Fatou's Lemma, for any positive constant C,

$$\begin{aligned}
\liminf_{k\to+\infty} \varepsilon_{c_k}^{-q_i} \int_{\mathbb{R}^N} V(\varepsilon_{c_k} x + \varepsilon_{c_k} y_{c_k}) |w_{c_k}(x)|^2 &\geq \int_{\mathbb{R}^N} \liminf_{k\to+\infty} \frac{V(\varepsilon_{c_k} x + \varepsilon_{c_k} y_{c_k})}{\varepsilon_{c_k}^{q_i}} |w_{c_k}(x)|^2 \\
&\geq \int_{\mathbb{R}^N} \liminf_{k\to+\infty} \frac{V(\varepsilon_{c_k} x + x_i)}{\varepsilon_{c_k}^{q_i}} |w_{c_k}(x + \frac{x_i - \varepsilon_{c_k} y_{c_k}}{\varepsilon_{c_k}})|^2 \\
&\geq \mu_i \int_{\mathbb{R}^N} \liminf_{k\to+\infty} |x|^{q_i} |w_{c_k}(x + \frac{x_i - \varepsilon_{c_k} y_{c_k}}{\varepsilon_{c_k}})|^2 \\
&\geq \mu_i \int_{B_1(0)} \liminf_{k\to+\infty} |x + \frac{\varepsilon_{c_k} y_{c_k} - x_i}{\varepsilon_{c_k}}|^{q_i} |w_{c_k}(x)|^2 \\
&\geq \frac{\mu_i \delta}{2} C.
\end{aligned}$$

Hence by (3.1) and (3.12),

$$\begin{aligned}
e_{c_k} &= \frac{1}{\varepsilon_{c_k}^2} \left(\frac{A(w_{c_k})}{2} - \frac{NB(w_{c_k})}{2(N+\alpha+2)} \right) + \frac{1}{2} \int_{\mathbb{R}^N} V(\varepsilon_{c_k} x + \varepsilon_{c_k} y_{c_k}) |w_{c_k}|^2 \\
&\geq \frac{1}{\varepsilon_{c_k}^2} \left[1 - \left(\frac{c}{c_*} \right)^{\frac{2(\alpha+2)}{N}} \right] + \frac{\mu_i \delta C}{4} \varepsilon_{c_k}^{q_i} \\
&\geq (1 + \frac{2}{q_i}) \left(\frac{q_i \delta \mu_i}{8} \right)^{\frac{2}{q_i+2}} \left[1 - \left(\frac{c_k}{c_*} \right)^{\frac{2(\alpha+2)}{N}} \right]^{\frac{q_i}{q_i+2}} C^{\frac{2}{q_i+2}} \\
&\geq (1 + \frac{2}{q_i}) \left(\frac{q_i \delta \mu_i}{8} \right)^{\frac{2}{q_i+2}} C^{\frac{2}{q_i+2}} \left[1 - \left(\frac{c_k}{c_*} \right)^{\frac{2(\alpha+2)}{N}} \right]^{\frac{q}{q+2}} \quad \text{as} \quad k \to +\infty,
\end{aligned} \tag{3.15}$$

which contradicts the upper bound obtained in **Step 1** since $C > 0$ is arbitrary. Then (3.14) holds. So for some $y_0 \in \mathbb{R}^N$,

$$\frac{\varepsilon_c y_c - x_{j_0}}{\varepsilon_c} \to y_0 \quad \text{and} \quad \varepsilon_c y_c \to x_{j_0} \quad \text{as } c \to (c_*)^-.$$

By the definition of $\{w_c\}$ and (3.12), $\{w_c\}$ is uniformly bounded in $H^1(\mathbb{R}^N)$. Then up to a subsequence, we may assume that for some $w_0 \in H^1(\mathbb{R}^N)$,

$$w_c \rightharpoonup w_0 \text{ in } H^1(\mathbb{R}^N), \qquad w_c \to w_0 \text{ in } L^s_{loc}(\mathbb{R}^N),\ 1 \leq s < 2^*$$

and

$$w_c(x) \to w_0(x) \text{ a.e. in } \mathbb{R}^N.$$

Then by (V_1) and the Fatou's Lemma again, there exists a constant $C_2 > 0$ independent of c such that

$$\begin{aligned}
&\liminf_{c\to(c_*)^-} \varepsilon_c^{-q_{j_0}} \int_{\mathbb{R}^N} V(\varepsilon_c x + \varepsilon_c y_c)|w_c(x)|^2 \\
\geq &\int_{\mathbb{R}^N} \liminf_{c\to(c_*)^-} \frac{V(\varepsilon_c x + \varepsilon_c y_c)}{|\varepsilon_c|^{q_{j_0}}}|w_c(x)|^2 \\
\geq &\int_{\mathbb{R}^N} \liminf_{c\to(c_*)^-} \frac{V(\varepsilon_c x + \varepsilon_c y_c)}{|\varepsilon_c x + \varepsilon_c y_c - x_{j_0}|^{q_{j_0}}}|x + \frac{\varepsilon_c y_c - x_{j_0}}{\varepsilon_c}|^{q_{j_0}}|w_c(x)|^2 \\
\geq &\mu_{j_0} \int_{B_1(0)} |x + y_0|^{q_{j_0}}|w_0(x)|^2 := C_2 > 0.
\end{aligned} \tag{3.16}$$

Similarly to (3.15), we have

$$e_c \geq (1 + \frac{2}{q_{j_0}}) \left(\frac{q_{j_0} C_2}{2}\right)^{\frac{2}{q_{j_0}+2}} \left[1 - \left(\frac{c}{c_*}\right)^{\frac{2(\alpha+2)}{N}}\right]^{\frac{q}{q+2}} := M_1 \left[1 - \left(\frac{c}{c_*}\right)^{\frac{2(\alpha+2)}{N}}\right]^{\frac{q}{q+2}}$$

as $c \to (c_*)^-$. □

Lemma 3.6. *Suppose that u_c is a minimizer of e_c and $V(x)$ satisfies $(V_0)(V_1)$, then there exist two positive constants $K_1 < K_2$ independent of c such that*

$$K_1 \left[1 - \left(\frac{c}{c_*}\right)^{\frac{2(\alpha+2)}{N}}\right]^{-\frac{2}{q+2}} \leq A(u_c) \leq K_2 \left[1 - \left(\frac{c}{c_*}\right)^{\frac{2(\alpha+2)}{N}}\right]^{-\frac{2}{q+2}} \quad \textit{as } c \to (c_*)^-.$$

Proof. The idea of the proof comes from that of Lemma 4 in [7], but it needs more careful analysis.

By (3.4), we see that

$$e_c = E(u_c) \geq \frac{1}{2}\left[1 - (\frac{c}{c_*})^{\frac{2(\alpha+2)}{N}}\right] A(u_c),$$

then by Lemma 3.5,

$$A(u_c) \leq 2M_2 \left[1 - \left(\frac{c}{c_*}\right)^{\frac{2(\alpha+2)}{N}}\right]^{-\frac{2}{q+2}},$$

where M_2 is given in Lemma 3.5.

For any fixed $b \in (0, c)$, there exist two functions $u_b \in \widetilde{S}(b)$, $u_c \in \widetilde{S}(c)$ such that $e_b = E(u_b)$ and $e_c = E(u_c)$ respectively. Then by (3.1), we see that

$$e_b \leq E\left(\frac{b}{c} u_c\right) < e_c + \frac{1}{2}\left[1 - \left(\frac{b}{c}\right)^{\frac{2(\alpha+2)}{N}}\right] A(u_c).$$

Let $\eta := \frac{c-b}{c_*-c} > 0$, then $\eta \to +\infty$ as $c \to (c_*)^-$. Then by Lemma 3.5, we have

$$\begin{aligned}
\frac{1}{2}A(u_c) &> \frac{e_b - e_c}{1-(\frac{b}{c})^{\frac{2(\alpha+2)}{N}}} \\
&\geq \frac{M_1(1-(\frac{b}{c_*})^{\frac{2(\alpha+2)}{N}})^{\frac{q}{q+2}} - M_2(1-(\frac{c}{c_*})^{\frac{2(\alpha+2)}{N}})^{\frac{q}{q+2}}}{1-(\frac{b}{c})^{\frac{2(\alpha+2)}{N}}} \\
&\geq \left[1-\left(\frac{c}{c_*}\right)^{\frac{2(\alpha+2)}{N}}\right]^{-\frac{2}{q+2}} \frac{M_1[\frac{1-(\frac{b}{c_*})^{\frac{2(\alpha+2)}{N}}}{1-(\frac{c}{c_*})^{\frac{2(\alpha+2)}{N}}}]^{\frac{q}{q+2}} - M_2}{(1-(\frac{b}{c})^{\frac{2(\alpha+2)}{N}})[1-(\frac{c}{c_*})^{\frac{2(\alpha+2)}{N}}]^{-1}} \\
&\geq \left[1-\left(\frac{c}{c_*}\right)^{\frac{2(\alpha+2)}{N}}\right]^{-\frac{2}{q+2}} \frac{M_1(\frac{N}{2(\alpha+2)})^{\frac{q}{q+2}}(1+\eta)^{\frac{q}{q+2}} - M_2}{\eta},
\end{aligned}$$

which gives the desired positive lower bound as $c \to (c_*)^-$. □

Proof of Theorem 1.5

Proof. Let $\{c_k\} \subset (0, c_*)$ be a sequence satisfying $c_k \to (c_*)^-$ as $k \to +\infty$ and denote $\{u_{c_k}\} \subset \widetilde{S}(c_k)$ to be a sequence of minimizers for e_{c_k}. Set

$$\varepsilon_k := [1-(\frac{c_k}{c_*})^{\frac{2(\alpha+2)}{N}}]^{\frac{1}{q+2}} > 0. \tag{3.17}$$

By (3.4), Lemmas 3.5 and 3.6, we see that

$$K_1\varepsilon_k^{-2} \leq A(u_{c_k}) \leq K_2\varepsilon_k^{-2}, \quad 0 \leq C(u_{c_k}) \leq 2M_2\varepsilon_k^q$$

Let

$$\tilde{w}_{c_k}(x) := \varepsilon_k^{\frac{N}{2}} u_{c_k}(\varepsilon_k x),$$

then $|\tilde{w}_c|_2 = c$ and

$$K_1 \leq A(\tilde{w}_{c_k}) \leq K_2, \qquad B(\tilde{w}_{c_k}) \leq \frac{N+\alpha+2}{N}K_2 \tag{3.18}$$

Let $\delta := \lim\limits_{k\to+\infty} \sup\limits_{y\in\mathbb{R}^N} \int_{B_1(0)} |\tilde{w}_{c_k}|^2$. If $\delta = 0$, then by the Vanishing Lemma 2.2, $\tilde{w}_{c_k} \to 0$ in $L^s(\mathbb{R}^N)$ as $k \to +\infty$, $2 < s < 2^*$. Hence by (1.5), $B(\tilde{w}_{c_k}) \to 0$. So

$$0 < \frac{K_1}{2} \leq \frac{A(\tilde{w}_{c_k})}{2} \leq e_{c_k}\varepsilon_k^2 + \frac{N}{2(N+\alpha+2)}B(\tilde{w}_{c_k}) \to 0 \text{ as } k \to +\infty,$$

which is a contradiction. Then $\delta > 0$ and there exists $\{y_k\} \subset \mathbb{R}^N$ such that $\int_{B_1(y_k)} |w_{c_k}|^2 \geq \frac{\delta}{2} > 0$. Set

$$w_{c_k}(x) := \tilde{w}_{c_k}(x + y_k) = \varepsilon_k^{\frac{N}{2}} u_{c_k}(\varepsilon_k x + \varepsilon y_k),$$

then
$$\int_{B_1(0)} |w_{c_k}|^2 \geq \frac{\delta}{2} > 0 \tag{3.19}$$
and
$$\int_{\mathbb{R}^N} V(\varepsilon_k x + \varepsilon_k y_k)|w_{c_k}(x)|^2 = C(u_{c_k}) \leq 2M_2\varepsilon_k^q. \tag{3.20}$$
Similar to the proof in Lemma 3.5, one can show that $\{\varepsilon_k y_k\}$ is uniformly bounded as $k \to +\infty$.

Since $u_{c_k} \in \widetilde{S}(c_k)$ is a minimizer of e_{c_k}, $(E|_{\widetilde{S}(c_k)})'(u_{c_k}) = 0$, i.e. there exists a sequence $\{\lambda_k\} \subset \mathbb{R}$ such that $E'(u_{c_k}) - \lambda_k u_{c_k} = 0$ in $\mathcal{H}^{-1}$, where $\mathcal{H}^{-1}$ denotes the dual space of $\mathcal{H}$. Then
$$\varepsilon_k^2\lambda_k = \frac{2\frac{N+\alpha+2}{N}\varepsilon_k^2 e_{c_k} - \frac{\alpha+2}{N}\varepsilon_k^2 C(u_{c_k}) - \frac{\alpha+2}{N}A(w_{c_k})}{c_k^2},$$
which and (3.18)(3.20) imply that there exists $\beta > 0$ such that
$$\varepsilon_k^2\lambda_k \to -\beta^2 \ \text{ as } k \to +\infty.$$
By the definition of w_{c_k}, we see that w_{c_k} satisfies the following equation
$$-\Delta w_{c_k} + \varepsilon_k^2 V(\varepsilon_k x + \varepsilon_k y_k)w_{c_k} - (I_\alpha * |w_{c_k}|^{\frac{N+\alpha+2}{N}})|w_{c_k}|^{\frac{N+\alpha+2}{N}-2}w_{c_k} = \lambda_k\varepsilon_k^2 w_{c_k} \ \text{ in } \mathbb{R}^N. \tag{3.21}$$
Since $\{w_{c_k}\}$ is uniformly bounded in $H^1(\mathbb{R}^N)$, there exists $w_0 \in H^1(\mathbb{R}^N)$ such that
$$w_{c_k} \rightharpoonup w_0 \ \text{ in } H^1(\mathbb{R}^N), \qquad w_{c_k} \to w_0 \ \text{ in } L^s_{loc}(\mathbb{R}^N),\ 1 \leq s < 2^*$$
and
$$w_{c_k}(x) \to w_0(x) \quad \text{a.e. in } \mathbb{R}^N.$$
Moreover, (3.19) implies that $w_0 \neq 0$. Then w_0 is a nontrivial solution of $-\Delta w_0 + \beta^2 w_0 = (I_\alpha * |w_0|^{\frac{N+\alpha+2}{N}})|w_0|^{\frac{N+\alpha+2}{N}-2}w_0$ in $\mathbb{R}^N$. Set
$$w_0(x) := \beta^{\frac{N}{2}} W_0(\beta x),$$
then W_0 is a nontrivial solution of
$$-\Delta W_0 + W_0 = (I_\alpha * |W_0|^{\frac{N+\alpha+2}{N}})|W_0|^{\frac{N+\alpha+2}{N}-2}W_0, \quad x \in \mathbb{R}^N. \tag{3.22}$$
Hence by Lemma 3.1 (2), we have $A(W_0) = \frac{N}{N+\alpha+2}B(W_0)$. So it follows from (3.1) that
$$c_*^{\frac{2(\alpha+2)}{N}} \leq \frac{\frac{N+\alpha+2}{N}A(W_0)|W_0|_2^{\frac{2(\alpha+2)}{N}}}{B(W_0)} = |W_0|_2^{\frac{2(\alpha+2)}{N}} = |w_0|_2^{\frac{2(\alpha+2)}{N}} \leq \lim_{k\to+\infty} |w_{c_k}|_2^{\frac{2(\alpha+2)}{N}} = c_*^{\frac{2(\alpha+2)}{N}},$$
i.e. $|w_0|_2 = |W_0|_2 = c_*$. Hence $w_{c_k} \to w_0$ in $L^2(\mathbb{R}^N)$ and then by the interpolation inequality,
$$w_{c_k} \to w_0 \ \text{ in } L^s(\mathbb{R}^N) \ \text{ for all } \ 2 \leq s < 2^*.$$
Moreover, Lemma 3.2 shows that W_0 is a groundstate solution of (3.22). So by Lemma 3.1 (3)(4), $W_0(x) = O(|x|^{-\frac{N-1}{2}}e^{-|x|})$ as $|x| \to +\infty$ and we may assume that up to translations, $W_0(x)$ is radially symmetric about the origin.

By (3.20), we see that for any $q_i \in \{q_1, \cdots, q_m\}$,

$$\frac{1}{\varepsilon^{q_i}} \int_{\mathbb{R}^N} V(\varepsilon_k x + \varepsilon_k y_k)|w_{c_k}(x)|^2 \le 2M_2.$$

Similarly to the proof of (3.14), there exists $x_{j_0} \in \{x_1, \cdots, x_m\}$ and $y_0 \in \mathbb{R}^N$ such that $\varepsilon_k y_k \to x_{j_0}$ and $\frac{\varepsilon_k y_k - x_{j_0}}{\varepsilon_k} \to y_0$ as $k \to +\infty$. Then similarly to (3.16), we see that

$$\begin{aligned}
\liminf_{k\to+\infty} \frac{1}{\varepsilon_k^q} \int_{\mathbb{R}^N} V(\varepsilon_k x + \varepsilon_k y_k)|w_{c_k}(x)|^2 &\ge \int_{\mathbb{R}^N} \liminf_{k\to+\infty} \frac{V(\varepsilon_k x + \varepsilon_k y_k)}{\varepsilon_k^{q_{j_0}}}|w_{c_k}(x)|^2 \\
&\ge \mu_{j_0} \int_{\mathbb{R}^N} |x + y_0|^{q_{j_0}} |w_0(x)|^2 \\
&= \frac{\mu_{j_0}}{\beta^{q_{j_0}}} \int_{\mathbb{R}^N} |x + \beta y_0|^{q_{j_0}} |W_0(|x|)|^2 \\
&\ge \frac{\mu_{j_0}}{\beta^{q_{j_0}}} \int_{\mathbb{R}^N} |x|^{q_{j_0}} |W_0(x)|^2,
\end{aligned} \tag{3.23}$$

where the last inequality is strict if and only if $y_0 \neq 0$. Hence similarly to (3.15),

$$\begin{aligned}
\liminf_{k\to+\infty} \frac{e_{c_k}}{\varepsilon_k^q} &\ge \frac{1}{2}A(w_0) + \liminf_{k\to+\infty} \frac{1}{2\varepsilon_k^q} \int_{\mathbb{R}^N} V(\varepsilon_k x + \varepsilon_k y_k)|w_{c_k}(x)|^2 \\
&\ge \frac{1}{2}\left(\beta^2 c_*^2 \frac{N}{\alpha+2} + \frac{\mu_{j_0}}{\beta^{q_{j_0}}} \int_{\mathbb{R}^N} |x|^{q_{j_0}} |W_0(x)|^2\right) \\
&\ge c_*^2 \left(\frac{\beta^2}{2}\frac{N}{\alpha+2} + \frac{\lambda_{j_0}^{q_{j_0}+2}}{q_{j_0}\beta^{q_{j_0}}}\right) \\
&\ge \frac{\lambda_{j_0}^2 c_*^2}{2}\left(\frac{N}{\alpha+2}\right)^{\frac{q_{j_0}}{q_{j_0}+2}} \frac{q_{j_0}+2}{q_{j_0}} \\
&\ge \frac{\lambda^2 c_*^2}{2}\left(\frac{N}{\alpha+2}\right)^{\frac{q}{q+2}} \frac{q+2}{q},
\end{aligned} \tag{3.24}$$

where $\lambda = \min\limits_{1\le i\le m} \lambda_i$ and $q = \max\limits_{1\le i\le m} q_i$.

On the other hand, for any $x_i \in \{x_1, \cdots, x_m\}$ and $t > 0$, let $v_k(x) = A_k \frac{c_k}{c_*}\left(\frac{t}{\varepsilon_k}\right)^{\frac{N}{2}} \varphi(x - x_i)W_0(\frac{t(x-x_i)}{\varepsilon_k})$, where φ is a cut-off function given as in (3.5) and $A_k > 0$ is chosen to satisfy that $v_k \in \widetilde{S}(c_k)$. Then $A_k \to 1$ as $k \to +\infty$. Similarly to (3.6), by the Dominated

Convergence theorem, we see that

$$\begin{aligned}\lim_{k\to+\infty}\frac{E(v_k)}{\varepsilon_k^q} &= \frac{t^2}{2}A(W_0)+\lim_{k\to+\infty}\frac{t^N}{2\varepsilon_k^{N+q}}\int_{\mathbb{R}^N}V(x)|\varphi(x-x_i)W_0(\frac{t(x-x_i)}{\varepsilon_k})|^2\\ &=\frac{1}{2}\left(\frac{t^2c_*^2N}{\alpha+2}+\frac{\overline{\mu}_i}{t^q}\int_{\mathbb{R}^N}|x|^q|W_0(x)|^2\right)\\ &\le c_*^2\left(\frac{t^2}{2}\frac{N}{\alpha+2}+\frac{\overline{\lambda}_i^{q+2}}{qt^q}\right),\end{aligned} \tag{3.25}$$

where

$$\overline{\mu}_i=\lim_{x\to x_i}\frac{V(x)}{|x-x_i|^q}=\begin{cases}\mu_i, & \text{if } q=q_i,\\ +\infty, & \text{if } q\neq q_i\end{cases}$$

and

$$\overline{\lambda}_i=\left(\frac{\overline{\mu}_iq}{2c_*^2}\int_{\mathbb{R}^N}|x|^q|W_0(x)|^2\right)^{\frac{1}{q+2}}=\begin{cases}\lambda_i, & \text{if } q=q_i,\\ +\infty, & \text{if } q\neq q_i\end{cases}.$$

So, since $t>0$ is arbitrary, by taking the infimum over $\{\overline{\lambda}_i\}_{i=1}^m$ in (3.25) and combining (3.24), we see that

$$\lim_{k\to+\infty}\frac{e_{c_k}}{\varepsilon_k^q}=\frac{\lambda^2c_*^2}{2}\left(\frac{N}{\alpha+2}\right)^{\frac{q}{q+2}}\frac{q+2}{q}.$$

Then (3.23)-(3.25) must be equalities, which imply that

$$y_0=0,\quad \beta=\left(\frac{\alpha+2}{N}\right)^{\frac{1}{q+2}}\lambda$$

and $\varepsilon_ky_k\to x_{j_0}\in\{x_i|\ \lambda_i=\lambda,\ 1\le i\le m\}$. Therefore,

$$\varepsilon_k^{\frac{N}{2}}u_{c_k}(\varepsilon_kx+\varepsilon_ky_k)=w_{c_k}(x)\to w_0(x)=\left((\frac{\alpha+2}{N})^{\frac{1}{q+2}}\lambda\right)^{\frac{N}{2}}W_0((\frac{\alpha+2}{N})^{\frac{1}{q+2}}\lambda x)$$

in $L^{\frac{2Ns}{N+\alpha}}(\mathbb{R}^N)$ for all $\frac{N+\alpha}{N}\le s<\frac{N+\alpha}{(N-2)_+}$.

□

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